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Solutions of the Equation of Helmholtz in an Angle with Vanishing Directional Derivatives Along each side. (I en II)

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Mei 1958

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Reprinted from Proceedings, Series A, 61, No. 4 and Indag. Math., 20, No. 4, 1958

MATHEMATICS

SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE WITH VANISHING DIRECTIONAL DERIVATIVES ALONG EACH SIDE. *) I

BY

D. VAN DANTZIG

(Communicated at the meeting of May 31, 1958)

1. *Introduction*

Within the context of a hydrodynamic research project [1]) the problem arose to obtain a function of Green $f(x, y, x_0, y_0)$ for the equation of Helmholtz

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-1\right) f=0 \tag{1.1}$$

defined in an angle A and having vanishing directional derivatives of constant direction along each side.

In polar coordinates r, φ the angle A can be represented by $\varphi_1<\varphi<\varphi_2$, $r>0$. The Helmholtz equation (1.1) takes the form

$$\left\{\left(r\frac{\partial}{\partial r}\right)^2+\frac{\partial^2}{\partial \varphi^2}-r^2\right\} f=0. \tag{1.2}$$

The required boundary conditions along the two sides $\varphi=\varphi_k$ $(k=1, 2)$ are

$$r \sin \gamma_k \frac{\partial f}{\partial r}-\cos \gamma_k \frac{\partial f}{\partial \varphi}=0. \tag{1.3}$$

The γ_k may have arbitrary complex, but constant, values. If γ_k is real it represents the angle in positive direction from the normal to the side $\varphi=\varphi_k$ of the angle A to the direction in which the derivative vanishes.

The following "hydrodynamic" cases are of particular interest:

a) $\gamma_1=\gamma_2$ b) $\cos\gamma_1 \cos\gamma_2=0$.

In this paper [2]) it will be assumed that γ_1 and γ_2 are real constants with $-\frac{1}{2}\pi \leq \gamma_k \leq \frac{1}{2}\pi$, $(k=1, 2)$.

*) Report T.W. 51 of the Mathematical Centre, Amsterdam, Netherlands.

[1]) Cf. D. VAN DANTZIG (1958).

[2]) I am indebted to Dr. H. A. LAUWERIER for his careful reading and in some places improving my original text. In particular his remark mentioned in § 4, and in consequence the replacement of the solution (5.9) by (5.10), (5.11) have been very clarifying. He also showed that by slightly altering the proofs my original assumption $0<\theta<\pi$ could be replaced by $0<\theta<2\pi$. In a forthcoming paper he also treats the limiting case $\theta=2\pi$.

The corresponding problems for a half plane ($\varphi_2-\varphi_1=\pi$, $\gamma_1=\gamma_2$) and a strip (cases a) and b)) have been solved in previous researches by G. W. VELTKAMP and by H. A. LAUWERIER (1953). The former has also determined the nature of the singularity of any solution of the present problem in the vertex $r=0$ of A in the abovementioned hydrodynamic cases.

It will sometimes be found useful to introduce complex coordinates in A:

$$z = re^{i\varphi} \stackrel{\text{def}}{=} x+iy \qquad \bar{z} = re^{-i\varphi} \stackrel{\text{def}}{=} x-iy \tag{1.4}$$

and the abbreviation

$$\theta \stackrel{\text{def}}{=} \varphi_2-\varphi_1. \tag{1.5}$$

We are especially interested in the solutions of (1.2), (1.3) which are regular, i.e. with continuous first and second partial derivatives in the closed angle $\bar{A}$ with the exception of a single point $z_0=r_0e^{i\varphi_0}$ in the open angle A in which

$$f = -(2\pi)^{-1}\ln|z-z_0|+0(1). \tag{1.6}$$

Summary. In § 2 it is shown that any regular solution $f(r, \varphi)$ of (1.1) can be represented as an integral transform of a harmonic function $U(\eta, \varphi)$. This result, in a slightly different form, is due to A. S. PETERS, who used it successfully for the problem of a sloping beach [3]). This harmonic function has the form $U_1(\eta+i\varphi)+U_2(-\eta+i\varphi)$, where U_1 and U_2 are sectionally holomorphic functions of their arguments, holomorphic in strips parallel to the real axis. In § 3 the boundary conditions are inserted. A special solution of (1.1) is obtained with $U_1(\zeta)=U_2(\zeta)=\phi(\zeta)$, where $\phi(\zeta)$ is meromorphic in the whole plane. For $\gamma_1>\gamma_2$ this solution is regular in $\bar{A}$; for $\gamma_1\leqq\gamma_2$ it has a singularity at the vertex $r=0$.

The function $\phi(\zeta)=\phi(\zeta;\gamma_1,\gamma_2)$ may be written as a product of the form $e(\zeta-i\varphi_1;\gamma_2)e(\zeta-i\varphi_2;-\gamma_1)$. A general class of functions of which $e(z,\gamma)$ is a specialization, is considered in § 6.

Representations in the form of an infinite integral and an infinite product are given and an asymptotic expression is derived. A general solution of (1.1) is obtained by putting $U_k(\zeta)=\phi(\zeta)P_k(\zeta)$ ($k=1$, $k=2$), where $P_k(\zeta)$ is a sectionally holomorphic function and periodic with period $2i\theta$. In § 4 the case of a single logarithmic singularity (1.6) is considered without taking any boundary conditions into account. It is shown that this implies the functions $U_k(\zeta)$ to be holomorphic in the strip $\varphi_1<\text{Im}\,\zeta<\varphi_2$ with the exception of a single line $\text{Im}\,\zeta=\varphi_0$, where $U_1(\zeta)$ and $U_2(\zeta)$ make prescribed jumps. In § 5 the results of § 3 and § 4 are combined. An explicit expression in the form of a double integral is obtained for a solution $f(r,\varphi)$ with a logarithmic singularity (r_0, φ_0).

For $\gamma_1\leqq\gamma_2$ this solution is unique. For $\gamma_1>\gamma_2$ a solution of the homogeneous problem may be added to the particular solution. The particular

[3]) Cf. also J. J. STOKER, Water waves, p. 95 et seq.

solution derived in § 5 has been adjusted to the supplementary condition of vanishing at the vertex $r=0$.

If this function of Green is denoted by $G(r, \varphi, r_0, \varphi_0, \gamma_1, \gamma_2)$ we have the symmetry relation

$$G(r, \varphi, r_0, \varphi_0, \gamma_1, \gamma_2) = G(r_0, \varphi_0, r, \varphi, -\gamma_1, -\gamma_2).$$

§ 2. *An integral transformation*

Let again $\bar{A}$ be $\{(r, \varphi)|\varphi_1 \leqq \varphi \leqq \varphi_2, r \geqq 0\}$ with $0<\theta<2\pi$ [4]), A its interior, S a finite point set in A. Let further $f(r, \varphi)$ be a complex valued function, continuous together with its (onesided) derivatives of first order in $\bar{A}-S$, and such that for some $\alpha>0, c>0$ uniformly in φ

$$f(r, \varphi) = 0(r^{-1-c}) \text{ for } r \to \infty, \quad \varphi_1 \leqq \varphi \leqq \varphi_2, \tag{2.1}$$

and

$$f(r, \varphi) = 0(|re^{i\varphi}-r_0e^{i\varphi_0}|^{-1+\alpha}) \text{ for } re^{i\varphi} \to r_0e^{i\varphi_0} \in S. \tag{2.2}$$

Then the Fourier transform

$$F(s, \varphi) \overset{\text{def}}{=} (2\pi)^{-1} \int_{-\infty}^{\infty} e^{irs} f(r, \varphi)\, dr \tag{2.3}$$

is holomorphic for $\operatorname{Im} s>0$, $\varphi_1 \leqq \varphi \leqq \varphi_2$, continuous for $\operatorname{Im} s \geqq 0$ and tending to zero for $\operatorname{Re} s \to \pm\infty$, $\operatorname{Im} s \geqq 0$. The inversion of (2.3)

$$f(r, \varphi) = \int_{-\infty}^{\infty} e^{-irs} F(s, \varphi)\, ds \tag{2.4}$$

holds for all $re^{i\varphi} \in \bar{A}-S$.

Now, let $f(r, \varphi)$ be $\in C_2$ in $A-S$ and satisfy the differential equation $(\Delta-1)f=0$, the condition (2.2) and for some $c>2$

$$\frac{\partial f}{\partial r} = 0(r^{-1-c}) \text{ for } r \to \infty, \quad \varphi_1 \leqq \varphi \leqq \varphi_2, \tag{2.5}$$

whence (2.1). Putting

$$U(\eta, \varphi) \overset{\text{def}}{=} \cosh \eta \cdot F(\sinh \eta, \varphi) \tag{2.6}$$

so that (2.4) becomes

$$f(r,\varphi) = \int_{-\infty}^{\infty} e^{-ir\sinh\eta}\, U(\eta, \varphi)\, d\eta, \tag{2.7}$$

then $U(\eta, \varphi)$ is a harmonic function in $B-S_1$, where

$$B \overset{\text{def}}{=} \{(\eta, \varphi)|\varphi_1 \leqq \varphi \leqq \varphi_2, -\infty<\eta<\infty\},$$

and

$$S_1 \overset{\text{def}}{=} \{(\eta, \varphi)\,|\,\exists_{r>0}\, re^{i\varphi} \in S\}.$$

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[4]) The condition $\theta < 2\pi$ could also be omitted if the solution is required to be unique with respect to (r, φ) only, not to (x, y), i.e. if A is an angle on a Riemann surface over the (x, y) plane having a branch point in $x = y = 0$.

The proof is as follows. For $(\eta, \varphi) \in B-S_1$, and $s=\sinh\eta$ we have

$$\frac{\partial^2 U}{\partial\eta^2}+\frac{\partial^2 U}{\partial\varphi^2}=\cosh\eta\left\{F+3\sinh\eta\frac{\partial F}{\partial s}+\cosh^2\eta\frac{\partial^2 F}{\partial s^2}+\frac{\partial^2 F}{\partial\varphi^2}\right\}\cdot$$

$$=(2\pi)^{-1}\cosh\eta\int_0^\infty e^{ir\sinh\eta}\left[\{1+3ir\sinh\eta-r^2\cosh^2\eta\}f(r,\varphi)+\frac{\partial^2 f}{\partial\varphi^2}\right]dr=$$

$$=(2\pi)^{-1}\cosh\eta\int_0^\infty e^{ir\sinh\eta}\left[\{1+3ir\sinh\eta-r^2\sinh^2\eta\}f(r,\varphi)-\left(r\frac{\partial}{\partial r}\right)^2 f\right]dr=$$

$$=(2\pi)^{-1}\cosh\eta\left[e^{ir\sinh\eta}\left\{(r+ir^2\sinh\eta)f-r^2\frac{\partial f}{\partial r}\right\}\right]_0^\infty=0$$

because of (2.5).

Consequently, putting

$$\zeta \overset{\text{def}}{=} \eta+i\varphi, \tag{2.8}$$

$U(\eta, \varphi)$ can be represented by means of two sectionally holomorphic functions U_1 and U_2 of ζ and $-\bar\zeta$ respectively which are holomorphic in each connected part of $B-S_1$. Thus

$$U(\eta,\varphi)=U_1(\zeta)+U_2(-\bar\zeta)=U_1(\eta+i\varphi)+U_2(-\eta+i\varphi). \tag{2.9}$$

Since the integral (2.4) vanishes for $r<0$ [5]), $U_1(\zeta)$ has no poles for $0<\operatorname{Im}\eta<\pi$ and $U_2(-\bar\zeta)$ none for $-\pi<\operatorname{Im}\eta<0$.

If, on the other hand, U is harmonic for $\varphi_1\leqq\varphi\leqq\varphi_2$, and if (2.7) holds for $r>0$ then f satisfies $(\Delta-1)f=0$.

The proof is as follows

$$(\Delta-1)f=-\int_{-\infty}^{\infty}e^{-ir\sinh\eta}\left[\sinh^2\eta\,U+\frac{i}{r}\sinh\eta\,U+\frac{1}{r^2}\frac{\partial^2 U}{\partial\eta^2}+U\right]d\eta=$$

$$=-\frac{1}{r^2}\left[e^{-ir\sinh\eta}\left(\frac{\partial U}{\partial\eta}+ir\cosh\eta\,U\right)\right]_{-\infty}^{\infty}=0$$

if $U\cosh\eta=o(1)$, whence $\partial U/\partial\eta=o(1)$.

If, however, more generally, $U(\cosh\eta)^c$ is bounded for some positive constant c as $\eta\to\pm\infty$ then (2.7) still may represent a solution of $(\Delta-1)f=0$ if the path of integration is shifted downwards over an arbitrarily small distance ε. In that case U should be harmonic in a somewhat larger region $\varphi_1-\varepsilon<\varphi<\varphi_2+\varepsilon$. Depending on the exact value of c the vertex $r=0$ may become a singular point.

Hence we have proved

Theorem 1. If, for $\varphi_1\leqq\varphi\leqq\varphi_2$, $r\geqq 0$ with the possible exception of a finite set S, the function $f(r,\varphi)\in C_2$ satisfies the equation of Helmholtz $(\Delta-1)f=0$ with the conditions (2.2) and (2.5), then it admits for $0<r<\infty$, $\varphi_1\leqq\varphi\leqq\varphi_2$ a representation of the form

$$f(r,\varphi)=\int_{-\infty}^{\infty}e^{-ir\sinh\eta}\{U_1(\eta+i\varphi)+U_2(-\eta+i\varphi)\}\,d\eta, \tag{2.10}$$

[5]) The point (r, φ) has to be distinghuished from $(-r, \varphi+\pi)$.

where $U_1(\zeta)$, $U_2(-\zeta)$ are sectionally holomorphic functions of their arguments for $\varphi_1 \leqq \operatorname{Im} \zeta \leqq \varphi_2$ which are holomorphic in any strip which does not contain a line $\varphi = \varphi_s$ where φ_s is the argument of a singular point $\in S$. In particular U_1 and U_2 may be chosen in such a way that the right-hand side of (2.10) vanishes for $r < 0$. If, on the other hand, (2.10) holds for $r > 0$, $\varphi_1 \leqq \varphi \leqq \varphi_2$ with $U(\eta, \varphi) = o(e^{-|\eta|})$ for $\eta \to \pm\infty$, $\varphi_1 \leqq \varphi \leqq \varphi_2$ where $U(\eta, \varphi) = U_1(\eta + i\varphi) + U_2(-\eta + i\varphi)$, then $f(r, \varphi)$ satisfies $(\Delta - 1)f = 0$ for $\varphi_1 \leqq \varphi \leqq \varphi_2$. The latter result holds also if the integral (2.10) is taken along a path in the complex η-plane passing from $\operatorname{Re} \eta = -\infty$ to $\operatorname{Re} \eta = +\infty$ such that $\operatorname{Im} \sinh \eta < 0$ and $U(\eta, \varphi) = 0(e^{c|\eta|})$ as $\operatorname{Re} \eta \to \pm\infty$.

§ 3. *Boundary conditions*

We now consider a function $f(r, \varphi)$ satisfying the homogeneous differential equation $(\Delta - 1)f = 0$ and the boundary conditions (1.3) and subject to the conditions of Theorem 1.

Inserting the boundary conditions into (2.10) we obtain for $k = 1, 2$

$$(3.1) \quad \int_{-\infty}^{\infty} e^{-ir\sinh\eta} \{\cosh(\eta - i\gamma_k)\, U_1(\eta + i\varphi_k) - \cosh(\eta + i\gamma_k)\, U_2(-\eta + i\varphi_k)\}\, d\eta = 0.$$

This holds primarily for $r > 0$. If, however, U_1 and U_2 are chosen such that the right-hand side of (2.10) vanishes for $r < 0$ or represents $Cf(|r|, \varphi)$ where C is an arbitrary constant, (we call this "condition V"), then (3.1) holds also for $r < 0$, hence

$$(3.2) \qquad \cosh(\eta - i\gamma_k)\, U_1(\eta + i\varphi_k) - \cosh(\eta + i\gamma_k)\, U_2(-\eta + i\varphi_k) = 0$$

for all real η. On the other hand (3.2) always entails (3.1).

Theorem 2. Under the conditions of theorem 1 the relations (3.2) are sufficient, and under condition V also necessary, in order that the boundary conditions (1.3) be fulfilled.

In order to solve the functional relations (3.2) we first determine a special solution with

$$(3.3) \qquad U_1(\zeta) = U_2(\zeta) = \phi(\zeta)$$

which is holomorphic in the whole strip $\varphi_1 \leqq \operatorname{Im} \zeta \leqq \varphi_2$.

If we put

$$(3.4) \qquad \phi(\zeta) = \frac{\phi_1(\zeta - i\varphi_2)}{\phi_2(\zeta - i\varphi_1)}, \quad \phi_1(-\zeta) = \phi_1(\zeta), \quad \phi_2(-\zeta) = \phi_2(\zeta)$$

then both $\phi_k(\zeta)$ have to satisfy an equation of the form

$$(3.5) \qquad \cosh(\zeta - i\gamma_k)\, \phi_k(\zeta - i\theta) = \cosh(\zeta + i\gamma_k)\, \phi_k(\zeta + i\theta).$$

In § 6 we shall obtain, as a specialisation of a more general class of functions, an analytic function $e(\zeta, \gamma)$ satisfying for a given constant θ the functional equation

$$(3.6) \qquad \frac{e(\zeta + i\theta, \gamma)}{e(\zeta - i\theta, \gamma)} = \frac{\cosh(\zeta + i\gamma)}{\cosh(\zeta - i\gamma)}$$

and such that

$$(3.7) \qquad e(-\zeta,\gamma) = e(\zeta,\gamma), \quad e(\zeta,-\gamma) = \{e(\zeta,\gamma)\}^{-1}.$$

This function has the following properties.

For all z and γ we have, assuming $\theta > 0$, a representation as an infinite product

$$(3.8) \qquad e(z,\gamma) = \prod_{m,n} \left\{1 + \frac{z^2}{(m\theta + \frac{1}{2}\pi n - \gamma)^2}\right\}\left\{1 + \frac{z^2}{(m\theta + \frac{1}{2}\pi n + \gamma)^2}\right\}^{-1}$$

where m, n run through the odd positive integers.

For $|\text{Im}\, z| < \theta + \frac{1}{2}\pi - |\gamma|$ we have

$$(3.9) \qquad \ln\, e(z,\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos tz}{t}\,\frac{\sinh \gamma t}{\sinh \theta t \sinh \frac{1}{2}\pi t}\,dt.$$

The asymptotic behaviour for $|\text{Re}\, z| \to \infty$ is as follows

$$(3.10) \qquad \ln\, e(z,\gamma) = \frac{\gamma}{\theta}\ln\,\cosh z + 0(1).$$

Substitution shows at once that

$$(3.11) \qquad \phi(\zeta) \overset{\text{def}}{=} e(\zeta - i\varphi_2, -\gamma_1)\, e(\zeta - i\varphi_1, \gamma_2)$$

is a solution of (3.2) and (3.3). To this we may add as a factor a periodic function of period $2i\theta$ which is symmetric with respect to $i\varphi_1$ and $i\varphi_2$. Thus the following general solution is obtained

$$(3.12) \qquad \phi^{(m)}(\zeta) \overset{\text{def}}{=} \{\cosh \pi\theta^{-1}(\zeta - i\varphi_1)\}^m \phi(\zeta)$$

where m is a positive integer.

According to (3.10) $\phi(\zeta)$ has the asymptotic behaviour

$$(3.13) \qquad \ln \phi(\zeta) = -\theta^{-1}(\gamma_1 - \gamma_2)\ln\,\cosh\,\zeta + 0(1).$$

We shall now suppose that $-\frac{1}{2}\pi \leqq \gamma_k \leqq \frac{1}{2}\pi\, (k=1,2)$.

Substitution of (3.12) into (2.10) gives

$$(3.14) \qquad f(r,\varphi) = \int_{-\infty}^{\infty} e^{-ir\sinh\eta}\{\phi^{(m)}(\eta + \varphi i) + \phi^{(m)}(-\eta + \varphi i)\}\,d\eta.$$

For $m \geqq 0$ this represents a solution of the homogeneous problem, regular in A with the possible exception of the vertex $r=0$. In view of (3.13) only for $\gamma_1 > \gamma_2$, $m=0$ a regular solution is obtained which for $r \to 0$ has a finite limit.

If now $f(r,\varphi)$ satisfies $(\Delta - 1)f = 0$ with the exception of a finite set S then the functions U_1 and U_2 are no longer holomorphic in the whole strip $\varphi_1 \leqq \text{Im}\,\zeta \leqq \varphi_2$. The solution of (3.2) is now obtained by introducing the sectionally holomorphic functions

$$(3.15) \qquad P_k(\zeta) \overset{\text{def}}{=} U_k(\zeta)/\phi_0(\zeta) \qquad k=1,2.$$

According to (3.2), (3.5) these functions must satisfy the conditions

$$(3.16) \qquad \begin{cases} P_1(\zeta + i\varphi_1) = P_2(-\zeta + i\varphi_1) \\ P_1(\zeta + i\varphi_2) = P_2(-\zeta + i\varphi_2) \end{cases}$$

for $\operatorname{Im} \zeta=0$. Defining $P_1(\zeta), P_2(\zeta)$ outside the strips adjacent to the boundaries by analytic continuation, the functional equations (3.16) hold also for complex ζ.

Hence we obtain by elimination of P_1

$$P_2(-\zeta+i\theta) = P_2(-\zeta-i\theta)$$

and similarly

$$P_1(\zeta+i\theta) = P_1(\zeta-i\theta)$$

i.e. P_1 and P_2 must be periodic with period $2i\theta$.

Because of (3.16), (3.15) becomes

$$\left\{\begin{array}{l} U_1(\zeta) = \phi_0(\zeta)\,P_1(\zeta) = \phi_0(\zeta)\,P_2(-\zeta+2i\varphi_1) \\ U_2(\zeta) = \phi_0(\zeta)\,P_2(\zeta) = \phi_0(\zeta)\,P_1(-\zeta+2i\varphi_2). \end{array}\right. \tag{3.17}$$

Hence we have proved

Theorem 3. Under the conditions of theorem 1 the boundary conditions (1.3) are satisfied if sectionally holomorphic functions $P_1(\zeta)$, $P_2(\zeta)$ exist, periodic with period $2i\theta$ such that the equalities (3.17) hold. Under condition V, moreover, the boundary conditions (1.3) imply the existence of $P_1(\zeta)$ and $P_2(\zeta)$.

§ 4. *A logarithmic singularity*

We now specialize to the case that S consists of one point $z_0=r_0e^{i\varphi_0} \in A$ only, and that $f(r, \varphi)$ has a logarithmic singularity (1.8) in z_0.

The fundamental solution of (1.1) having this singularity and regular in the whole plane except z_0 is given by the Bessel function $(2\pi)^{-1}K_0(|z-z_0|)$. According to its Laplace representation we have

$$K_0(\varrho) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-\varrho \cosh t}\, dt. \tag{4.1}$$

The path of integration along the real axis of the complex t-plane may be shifted within the strip $|\operatorname{Im} t| \leqq \frac{1}{2}\pi$. In particular with $t=w-\arg(z-z_0)$ we have

$$K_0(|z-z_0|) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp -\tfrac{1}{2}\{(z-z_0)\,e^{-w}+(\bar{z}-\bar{z}_0)\,e^{w}\}\, dw. \tag{4.2}$$

From this we may derive

$$K_0(|z-z_0|) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-ir\sinh \eta} \exp\{ir_0 \sinh(\eta+i|\varphi-\varphi_0|)\}\, d\eta. \tag{4.3}$$

Under the assumptions made $f(r,\varphi)-(2\pi)^{-1}K_0(|z-z_0|)$ must be regular everywhere in $\bar{A}$.

According to (2.10) $U_1(\zeta)$ and $U_2(\zeta)$ are holomorphic in the strip $\varphi_1 < \operatorname{Im} \zeta < \varphi_0$ and $\varphi_0 < \operatorname{Im} \zeta < \varphi_2$ and jump at $\operatorname{Im} \zeta=\varphi_0$. The magnitude of these jumps is determined by the standard solution $(2\pi)^{-1}K_0(|z-z_0|)$. This remains true if $(2\pi)^{-1}K_0(|z-z_0|)$ is replaced by any solution of (1.1) differing from it by a function everywhere regular in $\bar{A}$ [6]). In particular we may choose $(2\pi)^{-1}K_0(|z-z_0|) \pm (2\pi)^{-1}K_0(|z+z_0|)$ which leads to simpler formulae in the next section.

From (4.3) we may derive

$$(4.4) \qquad (2\pi)^{-1}K_0(|z-z_0|) = \int_{-\infty}^{\infty} e^{-ir\sinh\eta}\{V_1(\zeta)+V_2(-\bar{\zeta})\}\,d\eta$$

where

$$V_1(\zeta) = \begin{cases} 0 & \text{for } \varphi_1 \leqq \operatorname{Im}\zeta < \varphi_0 \\ (4\pi)^{-1}\exp ir_0\sinh(\zeta - i\varphi_0) & \text{for } \varphi_0 < \operatorname{Im}\zeta \leqq \varphi_2 \end{cases}$$

and

$$V_2(\zeta) = \begin{cases} (4\pi)^{-1}\exp -ir_0\sinh(\zeta - i\varphi_0) & \text{for } \varphi_1 \leqq \operatorname{Im}\zeta < \varphi_0 \\ 0 & \text{for } \varphi_0 < \operatorname{Im}\zeta \leqq \varphi_2. \end{cases}$$

Hence we obtain for $\varepsilon \searrow 0$

$$(4.5) \qquad \begin{cases} U_1\{\eta+i(\varphi_0+\varepsilon)\} - U_1\{\eta+i(\varphi_0-\varepsilon)\} \to (4\pi)^{-1}e^{ir_0\sinh\eta} \\ U_2\{\eta+i(\varphi_0+\varepsilon)\} - U_2\{\eta+i(\varphi_0-\varepsilon)\} \to -(4\pi)^{-1}e^{-ir_0\sinh\eta}. \end{cases}$$

Hence we have proved

Theorem 4. If $f(r,\varphi)$ satisfies the conditions of theorem 1, and if $f(r,\varphi)-(2\pi)^{-1}K_0(|z-z_0|)$ is bounded in $\bar{A}$, then the functions $U_1(\zeta)$ and $U_2(\zeta)$ of (2.10) are holomorphic in the strips $\varphi_1 < \operatorname{Im}\zeta < \varphi_0$ and $\varphi_0 < \operatorname{Im}\zeta < \varphi_2$ and jump at $\operatorname{Im}\zeta = \varphi_0$ according to (4.5). If on the other hand two such functions exist which are $o(e^{-|\eta|})$ for $\operatorname{Re}\zeta \to \pm\infty$, $\varphi_1 \leqq \operatorname{Im}\zeta \leqq \varphi_2$, then $f(r,\varphi)-(2\pi)^{-1}K_0(|z-z_0|)$ is bounded for $r \geqq 0$, $\varphi_1 \leqq \varphi \leqq \varphi_2$.

[6]) This remark is due to H. A. Lauwerier.

MATHEMATICS

SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE WITH VANISHING DIRECTIONAL DERIVATIVES ALONG EACH SIDE. II

BY

D. VAN DANTZIG

(Communicated at the meeting of May 31. 1958)

§ 5. *Boundary conditions and logarithmic singularity*

We now combine the results of § 3 and § 4. In order that the boundary conditions (1.3) be satisfied, U_1 and U_2 must be expressible in the form (3.18) by means of sectionally holomorphic functions $P_1(\zeta)$ and $P_2(\zeta)$ which are periodic with period $2i\theta$. In order that $f(r, \varphi)$ shall have the required singularity, U_1 and U_2 must make prescribed jumps at $\operatorname{Im} \zeta=\varphi_0$ according to (4.5).

Because of (3.17) this means for $P_1(\zeta)$ and $P_2(\zeta)$ for $\varepsilon \searrow 0$

$$P_1\{\eta+i(\varphi_0+\varepsilon)\}-P_1\{\eta+i(\varphi_0-\varepsilon)\} \to \frac{e^{ir_0 \sinh \eta}}{4\pi\,\phi(\eta+i\varphi_0)} \tag{5.1}$$

and

$$P_2\{\eta+i(\varphi_0+\varepsilon)\}-P_2\{\eta+i(\varphi_0-\varepsilon)\} \to \frac{e^{-ir_0 \sinh \eta}}{4\pi\,\phi(\eta+i\varphi_0)}. \tag{5.2}$$

According to Plemelj [7]) the sectionally holomorphic function

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt$$

makes a jump $f(z)$ at the real axis. Consequently the function

$$\frac{1}{4\theta i}\int_{-\infty}^{\infty} f(t)\coth\frac{\pi}{2\theta}(t-z)\,dt$$

makes jumps $f(\eta)$ at $\operatorname{Im} z=2m\theta i$, $\operatorname{Re} z=\eta$, where m is an integer. Moreover this function has the periodicity $2\theta i$. It is profitable to take $\varphi_1=0$ and $\varphi_2=\theta$. Then $P_1(\zeta)$ may assume the following form

$$P_1(\zeta) \overset{\text{def}}{=} \frac{1}{16\theta\pi i}\int_{-\infty}^{\infty}\frac{dt}{\phi(t+i\varphi_0)}\left\{\frac{e^{ir_0\sinh t}}{\tanh\frac{1}{2}\pi\theta^{-1}(t-\zeta+i\varphi_0)}-\frac{e^{-ir_0\sinh t}}{\tanh\frac{1}{2}\pi\theta^{-1}(t+\zeta+i\varphi_0)}\right\}, \tag{5.3}$$

valid for $\gamma_1 \leqslant \gamma_2$.

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[7]) Also cf. Muskhelishvili. Singular Integral Equations, p. 42.

By simple trigonometry (5.3) may be written in the form

$$(5.4)\quad \begin{cases} P_1(\zeta) = (16\theta\pi i)^{-1} \int\limits_{-\infty}^{+\infty} \dfrac{dt}{\phi(t+i\varphi_0)} \\ \qquad \left\{\dfrac{\cos(r_0 \sinh t)\sinh \pi\theta^{-1}\zeta + i \sin(r_0 \sinh t) \sinh \pi\theta^{-1}(t+i\varphi_0)}{\cosh \pi\theta^{-1}(t+i\varphi_0) - \cosh \pi\theta^{-1}\zeta}\right\}. \end{cases}$$

In a similar way, or by applying (3.17), we have

$$(5.5)\quad \begin{cases} P_2(\zeta) = (16\theta\pi i)^{-1} \int\limits_{-\infty}^{\infty} \dfrac{dt}{\phi(t+i\varphi_0)} \\ \qquad \left\{\dfrac{e^{ir_0 \sinh t}}{\tanh \frac{1}{2}\pi\theta^{-1}(t+\zeta+i\varphi_0)} - \dfrac{e^{-ir_0 \sinh t}}{\tanh \frac{1}{2}\pi\theta^{-1}(t-\zeta+i\varphi_0)}\right\}. \end{cases}$$

Substitution of (5.3), (5.4) into (3.8), (2.10) gives ultimately

$$(5.6)\quad f(r,\varphi) = (16\theta\pi i)^{-1} \int\limits_{-\infty}^{\infty} e^{-ir\sinh \eta}\, d\eta \int\limits_{-\infty}^{\infty} e^{ir_0 \sinh t}\, \psi(\eta, t)\, dt$$

where

$$(5.7)\quad \begin{cases} \psi(\zeta, s) \overset{\text{def}}{=} \dfrac{\phi(\zeta)}{\phi(s)} \coth \frac{1}{2}\pi\theta^{-1}(s-\zeta) + \dfrac{\phi(-\bar\zeta)}{\phi(-\bar s)} \coth \frac{1}{2}\pi\theta^{-1}(\bar s - \bar\zeta) + \\ \qquad + \dfrac{\phi(\zeta)}{\phi(-\bar s)} \coth \frac{1}{2}\pi\theta^{-1}(\bar s - \zeta) + \dfrac{\phi(-\bar\zeta)}{\phi(s)} \coth \frac{1}{2}\pi\theta^{-1}(s-\bar\zeta), \end{cases}$$

with $\zeta = \eta + i\varphi$ and $s = t + i\varphi_0$.

This solution may also be written in the form

$$(5.8)\quad \begin{cases} f(r,\varphi) = (8\theta\pi)^{-1} \int\limits_{-\infty}^{\infty} d\eta \int\limits_{-\infty}^{\infty} dt\, \dfrac{\phi(\zeta)}{\phi(s)} \\ \qquad \left\{\dfrac{\sin(r\sinh\eta + r_0 \sinh t)}{\tanh \frac{1}{2}\pi\theta^{-1}(s+\zeta)} - \dfrac{\sin(r\sinh\eta - r_0\sinh t)}{\tanh \frac{1}{2}\pi\theta^{-1}(s-\zeta)}\right\}. \end{cases}$$

This again may be brought in the form

$$(5.9)\quad \begin{cases} f(r,\varphi) = (4\theta\pi)^{-1} \int\limits_{-\infty}^{\infty} d\eta \int\limits_{-\infty}^{\infty} dt\, \dfrac{\phi(\zeta)}{\phi(s)} \left[\dfrac{\sinh \pi\theta^{-1}s \cos(r\sinh\eta)\sin(r_0\sinh t)}{\cosh \pi\theta^{-1}s - \cosh\pi\theta^{-1}\zeta} - \right. \\ \qquad \left. - \dfrac{\sinh \pi\theta^{-1}\zeta \sin(r\sinh\eta)\cos(r_0\sinh t)}{\cosh\pi\theta^{-1}s - \cosh\pi\theta^{-1}\zeta}\right]. \end{cases}$$

According to the remark made in the previous section this expression may be simplified in the following way. If r_0 is replaced by $-r_0$ the right-hand side of (5.9) represents a solution of $(\Delta - 1)f = 0$ satisfying the boundary conditions which is regular in A. By addition and subtraction we obtain resp.

$$(5.10)\quad \begin{cases} f_1(r,\varphi) = \dfrac{1}{2\theta\pi} \int\limits_{-\infty}^{\infty} d\eta \int\limits_{-\infty}^{\infty} dt\, \dfrac{\phi(\zeta)}{\phi(s)} \\ \qquad \cos(r\sinh\eta)\sin(r_0\sinh t)\, \dfrac{\sinh\pi\theta^{-1}s}{\cosh\pi\theta^{-1}s - \cosh\pi\theta^{-1}\zeta} \end{cases}$$

and

$$(5.11)\quad \begin{cases} f_2(r,\varphi) = \frac{1}{2\theta\pi}\int\limits_{-\infty}^{\infty} d\eta \int\limits_{-\infty}^{\infty} dt\, \frac{\phi(\zeta)}{\phi(s)} \\ \qquad\qquad \sin(r\sinh\eta)\cos(r_0\sinh t)\, \frac{\sinh\pi\theta^{-1}\zeta}{\cosh\pi\theta^{-1}\zeta-\cosh\pi\theta^{-1}s}. \end{cases}$$

The former expression gives Green's function in the case $\gamma_1 \leqq \gamma_2$. The latter expression gives a function of Green in the case $\gamma_1 > \gamma_2$. This function is not unique but (5.11) represents that Green's function that vanishes at $r=0$. Both expressions (5.10) and (5.11) are clearly convergent with respect to η and t. Using the fact that $\{\phi(\zeta)\}^{-1}$ is obtained from $\phi(\zeta)$ by changing the signs of both γ_1 and γ_2 we notice the symmetry between (5.10) and (5.11) which are transformed into each other by $r, \varphi, \gamma_1, \gamma_2 \leftrightarrow r_0, \varphi_0, -\gamma_1, -\gamma_2$.
Hence we have proved:

Theorem 5. Let $f(r, \varphi)$ satisfy the conditions of Theorem 1, the boundary conditions (1.3) and let $f(r, \varphi)-(2\pi)^{-1}K_0(|z-z_0|)$ be bounded in $\bar{A}$. Then, if $\gamma_1 \leqq \gamma_2$, $f(r, \varphi)$ is uniquely determined and may be represented by (5.10). If $\gamma_1 > \gamma_2$, $f(r, \varphi)$ is not uniquely determined. However, under the condition of vanishing at the vertex $r=0$, it is unique and may be represented by (5.11).

With due observance of this supplementary condition the following skew symmetry relation holds

$$f(r, \varphi, r_0, \varphi_0, \gamma_1, \gamma_2) = f(r_0, \varphi_0, r, \varphi, -\gamma_1, -\gamma_2).$$

§ 6. *Appendix*

In this section a general class of functions E_k will be considered from which the function $e(\zeta, \gamma)$ which is used in section 3, and which for a given constant θ satisfies the functional relation

$$(6.1)\qquad \frac{e(\zeta+i\theta,\gamma)}{e(\zeta-i\theta,\gamma)} = \frac{\cosh(\zeta+i\gamma)}{\cosh(\zeta-i\gamma)},$$

may be derived by specialization.

We define for $k \geqq 0$

$$(6.2)\qquad \wedge_k \begin{pmatrix} p_0, \ldots, p_k \\ q_0, \ldots, q_k \end{pmatrix} \overset{\text{def}}{=} \tfrac{1}{2} \int\limits_{-\infty}^{\infty} dt \prod_0^k \frac{\sinh p_j t}{\sinh q_j t}$$

for any set (p_j, q_j) for which the integral converges. With small loss of generality we may assume that all $\operatorname{Re} q_j$ are positive. We define

$$(6.3)\qquad \mu \overset{\text{def}}{=} \min \operatorname{Re} q_j > 0.$$

Moreover the $\wedge_k$ are invariant under arbitrary permutations of the p_j as well as of the q_j. If one of the p_j equals one of the q_j, say $p_k = q_k$, we have obviously a reduction of the following kind

$$(6.4)\qquad \wedge_k \begin{pmatrix} p_0, \ldots, p_{k-1}, z \\ q_0, \ldots, q_{k-1}, z \end{pmatrix} = \wedge_{k-1} \begin{pmatrix} p_0, \ldots, p_{k-1} \\ q_0, \ldots, q_{k-1} \end{pmatrix}.$$

As yet the $\wedge_k$ are defined under the restriction

$$(6.5) \qquad \sum_0^k |\mathrm{Re}\, p_j| < \sum_0^k \mathrm{Re}\, q_j.$$

However, the region of definition can be extended by expansion of the integrand of 6.2 into exponentials, putting $\frac{1}{2}\int_{-\infty}^{\infty} = \int_0^{\infty}$.

Introducing the abbreviations

$$(6.6) \qquad \begin{cases} P_\varepsilon = P_{\varepsilon_0,\dots,\varepsilon_k} \overset{\text{def}}{=} \sum_0^k \varepsilon_j p_j \\ Q_n = Q_{n_0,\dots,n_k} \overset{\text{def}}{=} \sum_0^k (2n_j+1) q_j \end{cases}$$

where $n_0, \dots, n_k$ run independently through the non-negative integers, and $\varepsilon_0, \dots, \varepsilon_k$ through the pair $-1, +1$, we have

$$(6.7) \qquad \begin{cases} 2^{k+1} \prod_0^k \sinh p_j t = \sum_\varepsilon \prod_0^k \varepsilon_j \exp P_\varepsilon t \\ 2^{-k-1} \prod_0^k (\sinh q_j t)^{-1} = \sum_n \exp -Q_n t, \end{cases}$$

whence [8])

$$(6.8) \qquad \wedge_k \begin{pmatrix} p_0, \dots, p_k \\ q_0, \dots, q_k \end{pmatrix} = \sum_n \sum_\varepsilon (Q_n - P_\varepsilon)^{-1} \prod \varepsilon_j.$$

Since

$\sum_\varepsilon (\prod \varepsilon_j) \cdot P_\varepsilon^l = 0$ for $l = 0, 1, \dots k$, $\sum_\varepsilon (Q_n - P_\varepsilon)^{-1} \prod \varepsilon_j$ is $0((\sum n_j)^{-k-2})$

at infinity, so that the multiple sum on the right-hand side of 6.8 is absolutely convergent as soon as no denominator vanishes. Thus 6.8 gives the continuation of 6.2 where the latter does not exist.

Further we define for $k \geqq 0$

$$(6.9) \qquad E_k(z) = E_k \begin{pmatrix} z; p_1, \dots, p_k \\ q_0, q_1, \dots, q_k \end{pmatrix} \overset{\text{def}}{=} \exp \left[-\int_0^z \wedge_k \begin{pmatrix} p, p_1, \dots, p_k \\ q_0, q_1, \dots, q_k \end{pmatrix} dp \right]$$

so that $-\wedge_k$ is the logarithmic derivative of $E_k(p_0)$. Evidently

$$(6.10) \qquad \ln E_k \begin{pmatrix} z; p_1, \dots, p_k \\ q_0, q_1, \dots, q_k \end{pmatrix} = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{t} (\cosh tz - 1) \frac{\prod^* \sinh p_j t}{\prod \sinh q_j t}$$

for those z for which the right-hand side exists.

If $\varkappa \overset{\text{def}}{=} \sum_0^k \mathrm{Re}\, q_j - \sum_1^k |\mathrm{Re}\, p_j| > 0$ then 6.10 holds in the strip $|\mathrm{Re}\, z| < \varkappa$.

From 6.8 we obtain the continuation

$$(6.11) \qquad E_k \begin{pmatrix} z; p_1, \dots, p_k \\ q_0, q_1, \dots, q_k \end{pmatrix} = \prod_n \prod_\varepsilon \left[1 - \frac{z^2}{(Q_n - P_\varepsilon)^2} \right]^{\Pi^* \varepsilon_j}$$

where now ε_j runs through $\varepsilon_1, \dots, \varepsilon_k$.

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[8]) In what follows $\prod$ is an abbreviation for $\prod_{j=0}^k$ and $\prod^*$ for $\prod_{j=1}^k$.

The points $z=\pm(Q_n-P_s)$ are poles if $\prod^* \varepsilon_j=-1$, zeros if $\prod^* \varepsilon_j=+1$. We note the particular cases

$$\wedge_0\binom{p}{q}=\frac{\pi}{2q}\operatorname{tg}\frac{\pi p}{2q} \tag{6.12}$$

and

$$E_0\binom{z}{q}=\cos\frac{\pi z}{2q}=\prod_{n=0}^{\infty}\left(1-\frac{z^2}{(2n+1)^2q^2}\right). \tag{6.13}$$

From 6.2 the following difference equation may be obtained

$$\left\{\begin{aligned}&\wedge_k\begin{pmatrix}p_0+q_0, p_1, \ldots, p_k\\ q_0 \quad\ , q_1, \ldots, q_k\end{pmatrix}-\wedge_k\begin{pmatrix}p_0-q_0, p_1, \ldots, p_k\\ q_0 \quad\ , q_1, \ldots, q_k\end{pmatrix}=\\ &=\wedge_{k-1}\begin{pmatrix}p_0+p_1, p_2, \ldots, p_k\\ q_1 \quad\ , q_2, \ldots, q_k\end{pmatrix}-\wedge_{k-1}\begin{pmatrix}p_0-p_1, p_2, \ldots, p_k\\ q_1 \quad\ , q_2, \ldots, q_k\end{pmatrix}.\end{aligned}\right. \tag{6.14}$$

According to the principle of continuation this holds everywhere with the exception of an enumerable set.

Similarly

$$\left\{\begin{aligned}&E_k\begin{pmatrix}z+q_0; p_1, \ldots, p_k\\ q_0 \quad\ , q_1, \ldots, q_k\end{pmatrix}\Big/ E_k\begin{pmatrix}z-q_0; p_1, \ldots, p_k\\ q_0 \quad\ , q_1, \ldots, q_k\end{pmatrix}=\\ &=E_{k-1}\begin{pmatrix}z+p_1; p_2, \ldots, p_k\\ q_1 \quad\ , q_2, \ldots, q_k\end{pmatrix}\Big/ E_{k-1}\begin{pmatrix}z-p_1; p_2, \ldots, p_k\\ q_1 \quad\ , q_2, \ldots, q_k\end{pmatrix}.\end{aligned}\right. \tag{6.15}$$

Taking in particular $k=1$, $q_1=\frac{1}{2}\pi$ we obtain in view of 6.13

$$E_1\begin{pmatrix}z+q, p\\ q \ , \frac{1}{2}\pi\end{pmatrix}\Big/ E_1\begin{pmatrix}z-q, p\\ q \ , \frac{1}{2}\pi\end{pmatrix}=\cos(z+p)/\cos(z-p) \tag{6.16}$$

which is the identity underlying 6.1. We need only take

$$e(z,\gamma)\overset{\text{def}}{=}E_1\begin{pmatrix}-iz, \gamma\\ \theta \ , \frac{1}{2}\pi\end{pmatrix}. \tag{6.17}$$

If $\operatorname{Im}(q_1/q_0)\neq 0$, $E_1(z)$ is the quotient of two products of two double gamma functions each, having their poles in opposite angles, viz. $z=Q+p$ and $z=-Q-p$ for the numerator, $z=Q-p$ and $z=-Q+p$ for the denominator.

Hence the pattern of poles of the numerator is obtained from that of the denominator by translations over $\pm 2p$. In our application 6.17 q_1/q_0 is real and positive. Then the pattern of poles of $e(z,\gamma)$ degenerates into countable sets on lines parallel to the imaginary axis viz.

$$z=\pm i(m\theta+\tfrac{1}{2}\pi n\mp\gamma)$$

where m, n run through odd positive integers.

We further need some knowledge of the asymptotic behaviour of E_k for $\eta=\operatorname{Im} z\to\pm\infty$. We first consider E_k in the strip of convergence

$|\text{Re } z| < \varkappa$ of the integral representation 6.10 only. Then it is easily seen that $E_k = 0(|\eta|)$ for $\eta \to \pm\infty$, and, more precisely, that

$$\lim_{|\eta|\to\infty} |\eta|^{-1} \ln E_k \begin{pmatrix} \xi+i\eta\,; p_1, \ldots, p_k \\ q_0 \quad, q_1, \ldots, q_k \end{pmatrix} = \tfrac{1}{2}\pi \prod{}^* p_j / \prod q_j. \tag{6.18}$$

We restrict z to the strip $|\text{Re } z| < \varkappa' \overset{\text{def}}{=} \min(\text{Re } p_0, \varkappa)$ where p_0 is some constant with $\text{Re } p_0 > 0$. Then it follows from 6.18 and the corresponding relation for $k=0$ that

$$\ln E_k \begin{pmatrix} \xi+i\eta\,; p_1, \ldots, p_k \\ q^0 \quad, q_1, \ldots, q_k \end{pmatrix} - \prod_0^k \frac{p_j}{q_j} \ln E_0 \begin{pmatrix} \xi+i\eta \\ p_0 \end{pmatrix} = o(|\eta|). \tag{6.19}$$

Actually it has a limit for $|\eta| \to \infty$ which can easily be determined as follows.

The left-hand side of 6.19 equals $-\frac{1}{2}\int_{-\infty}^{\infty} \{\cosh(\xi+i\eta)t - 1\} g(t)dt$ if we define

$$g(t) \overset{\text{def}}{=} \frac{1}{t \sinh p_0 t} \left\{ \prod_0^k \frac{\sinh p_j t}{\sinh q_j t} - \prod_0^k \frac{p_j}{q_j} \right\}. \tag{6.20}$$

Since $g(t)$ is regular in the strip $|\text{Im } t| < \lambda$ where

$$\lambda \overset{\text{def}}{=} \min\left(\text{Re}\frac{\pi}{p_0}, \text{Re}\frac{\pi}{q_j}\right) \quad j = 0, \ldots, k,$$

we have

$$\int_{-\infty}^{\infty} \cosh(\xi+i\eta)t\, g(t)\, dt = 0(e^{-\lambda'|\eta|}) \text{ for any } \lambda' < \lambda. \tag{6.21}$$

Thus we have proved

Theorem 6. If $k \geqq 0$, $\min\{\text{Re } p_0, \text{Re } q_0, \text{Re } q_1, \ldots, \text{Re } q_k\} > 0$,

$$\varkappa \overset{\text{def}}{=} \sum_0^k \text{Re } q_j - \sum_1^k |\text{Re } p_j| > 0,$$

then for $|\text{Re } z| < \min(\text{Re } p_0, \varkappa)$ and $|\text{Im } z| \to \infty$,

$$\ln E_k \begin{pmatrix} z\,; p_1, \ldots, p_k \\ q_0, q_1, \ldots, q_k \end{pmatrix} - \prod_0^k \frac{p_j}{q_j} \ln E_0 \begin{pmatrix} z \\ p_0 \end{pmatrix} =$$

$$\tfrac{1}{2} \int_{-\infty}^{\infty} \prod_0^k \left\{ \frac{\sinh p_j t}{\sinh q_j t} - \frac{p_j}{q_j} \right\} \frac{dt}{t \sinh p_0 t} + 0(e^{-\lambda'|\text{Im } z|})$$

where

$$\lambda' < \lambda \overset{\text{def}}{=} \min \text{Re}\left(\frac{\pi}{p_0}, \frac{\pi}{q_0}, \frac{\pi}{q_1}, \ldots, \frac{\pi}{q_k}\right).$$

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